Using Matlab to Numerically Solve Prey-Predator Models with Diffusion

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The Lotka-Volterra equations are commonly used to describe the dynamics of the interaction between two species, one as a predator and one as a prey. To make the model more realistic, we modify it so that the prey species exhibits logistic growth rather than exponential growth. We also add in terms that allow both populations to disperse from their initial location. Through numerical analysis via Matlab, we simulate the outcome of such modifications.

Keywords: Lotka-Volterra model, Diffusion, Finite Forward Difference Method, Matlab

The Lotka-Volterra model is a pair of differential equations that describe a simple case of predator-prey (or parasite-host) dynamics. These equations were derived independently by Alfred Lotka [6] and Vito Volterra [11] in the mid 1920's.

The assumptions of the model in its most basic form are as follows:

- 1. The prey always finds enough food to sustain itself and grow exponentially when the predatory is absent.
- 2. The food supply of the predator population depends entirely on the size of the prey population. In other words, the predators will not switch to another type of prey.
- 3. The predators have an unlimited appetite.
- 4. The rate of change of the populations are proportional to their respective sizes.
- 5. No environmental changes that favor one of the populations occur. Genetic adaptation is inconsequential.

Given the above-mentioned assumptions, the set of differential equations representing the model is given by

$$\begin{cases} \frac{\partial U}{\partial t} = AU - BUV\\ \frac{\partial V}{\partial t} = CUV - DV \end{cases}$$

where U(t) is the number of prey at time t, V(t) is the predators at time t, A is the natural growth rate of prey in the absence of predators, B is the rate of prey loss due to predator/prey interaction, C is the growth of predators due to predator/prey interaction, and D is the rate of predator loss due to natural death or immigration. A, B, C, and D are positive constants. The system has two equilibrium points, (U, V) = (0, 0) (extinction) and (U, V) = (D/C, A/B) (coexistence) [7].

Rosenzweig and McArthur [9] would later extend the model to include three-species interaction, prey hiding places, and density dependent prey growth. Arditi and Ginzburg argued that predator-prey interactions should not be treated as random occurences and that the predator's consumption rate should depend on the ratio of prey and predator densities and not simply the prey density alone [1]. The Lotka-Volterra equations have also been used in economic theory, where "predators" and "prey" have taken on roles of economic parameters such as prices and outputs of goods [3, 4, 8].



Figure 1: A typical solution of the Lotka-Volterra equations. The graph of the prey (U) is in red and the graph of the predators (V) is in blue. The coefficients used were A = 1.5, B = 1, C = 1, D = 3. Source: Wolfram MathWorld http://mathworld.wolfram.com/Lotka-VolterraEquations.html [12]

The basic model is unrealistic for a few reasons. First, it can be shown that coexistence equilibrium point is not stable. Instead, the prey and predator populations cycle repeatedly without ever settling (see Figure 1), and while this cyclic behavior has been observed in nature, it is not common (http://www.stolaf.edu/people/mckelvey/envision.dir/lotka-volt.html). One key improvement on the Lotka-Volterra models is the incorporation of a diffusion effect. Takeuchi [10] analyzed the diffusion effect on the stability of Lotka-Volterra systems, and Hastings [5] derived conditions for global stability of Lotka-Volterra systems with diffusion. Next, it does not consider any competition among prey or predators, and thus, prey population may grow infinitely without any resource limits. Exponential growth of a population cannot continue indefinitely.

The goal of this paper is to come up with a more realistic version of the Lotka-Volterra model and to provide a tool that allows researchers to explore dynamics of spatiotemporal dynamics of Lotka-Volterra models with diffusion. We consider a modified system with logistic growth of the prey. We also allow both predator and prey to disperse by diffusion. Then, solutions of the model will be estimated using a finite forward difference scheme under varying initial population distributions and dispersion rates.

The modified model is

$$\begin{cases} \frac{\partial U}{\partial t} = AU\left(1 - \frac{U}{K}\right) - BUV + D_1 \nabla^2 U\\ \frac{\partial V}{\partial t} = CUV - DV + D_2 \nabla^2 V \end{cases}$$

where K > 0 is the prey carrying capacity and D_1 and D_2 are the diffusion constants. We nondimensionalize the system by using

$$u = \frac{U}{K}, \quad v = \frac{BV}{A}, \quad t^* = At, \quad x^* = x \left(\frac{A}{D_2}\right)^{1/2}$$
$$D^* = \frac{D_1}{D_2}, \quad a = \frac{CK}{A}, \quad b = \frac{D}{CK}$$

Considering only the one-dimensional problem, and dropping the asterisks for notational simplicity:

$$\begin{cases} \frac{\partial u}{\partial t} = u(1 - u - v) + D \cdot \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial v}{\partial t} = av(u - b) + \frac{\partial^2 v}{\partial t^2} \\ 0 < x < L, \ t > 0 \end{cases}$$

It is easy to check that (u, v) = (b, 1-b) is a non-trivial solution to the model. Also, note as $u = \frac{U}{K}$, 0 < u < 1. If we assume that the net flux at the boundaries is zero, then the boundary conditions are

$$\frac{\partial u}{\partial x}\Big|_{(t,0)} = 0 \qquad \frac{\partial v}{\partial x}\Big|_{(t,0)} = 0 \qquad \frac{\partial u}{\partial x}\Big|_{(t,L)} = 0 \qquad \frac{\partial v}{\partial x}\Big|_{(t,L)} = 0$$

Numerical methods used for analysis

To approximate the solutions of the system, we use a finite-difference method. The domain of the model is partitioned in time using a mesh $t_0, t_1, ..., t_N$ and in space using a mesh $x_0, x_1, ..., x_J$. We use a uniform partition for both, so the difference between two consecutive time points will be Δt and between two consecutive space points will be Δx . The point u_j^n will be the approximation of u at location j and at time n. The same is true for v_j^n . Of the three common methods for approximating solutions using a finite difference, we have chosen the forward difference method. This method was chosen because it is an explicit method of determining solutions. Estimated values of u_j^{n+1} and v_j^{n+1} can be computed as a function of their respective values at time step n. To estimate the derivative term $\frac{\partial u}{\partial t}$, we use

$$\frac{\partial u}{\partial t} \approx \frac{u_j^{n+1} - u_j^n}{\Delta t} \,,$$

and to estimate the dispersion term $\frac{\partial^2 u}{\partial r^2}$, we use

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

Thus, the first equation of the model can be rewritten as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = u_j^n (1 - u_j^n - v_j^n) + D \cdot \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

The second equation may be rewritten in a similar manner. Solving for u_j^{n+1} and v_j^{n+1} respectively, the scheme for the finite forward-difference method is

$$u_{j}^{n+1} = D\left(\frac{\Delta t}{(\Delta x)^{2}}\right)\left(u_{j-1}^{n} - 2u_{j}^{n} + u_{j+1}^{n}\right) + \Delta t u_{j}^{n}\left(1 - u_{j}^{n} - v_{j}^{n}\right) + u_{j}^{n}$$
$$v_{j}^{n+1} = \left(\frac{\Delta t}{(\Delta x)^{2}}\right)\left(v_{j-1}^{n} - 2v_{j}^{n} + v_{j+1}^{n}\right) + a\Delta t v_{j}^{n}\left(u_{j}^{n} - b\right) + v_{j}^{n}$$

for $2 \notin j \notin N_x$ - 2. For the mesh points next to the boundary, we use $u_1^n = u_2^n$, $u_{N_x-1}^n = u_{N_x-2}^n$, $v_1^n = v_2^n$, and $v_{N_x-1}^n = v_{N_x-2}^n$.

Using Matlab (see Appendix for code), I tested the model under varying parameter values and initial conditions. I assume both populations have a normal distribution on the interval [0, 1] In Figure 1, the prey population has a large population most concentrated at x = 0.2 and that the predators have a smaller population most concentrated at x = 0.7. Thus,

 $u^{0}(x) = 0.8 \exp(-50(x-0.2)^{2})$ and $v^{0}(x) = 0.3 \exp(-50(x-0.7)^{2})$. The graphs show the initial distribution along with the distribution at t = 300, t = 1000, and t = 10000 time steps. Other parameter values are a = 0.1, b = 1, and D = 0.5.



Figure 2: Numerical approximations of the modified Lotka-Volterra model with a larger population of prey and a smaller population of predators.. Figure 2a shows the distribution of the population of the prey species at various time steps, and Figure 2b shows the distribution of population of the predator species at various time steps.

Next, I assumed a lower population of prey and a higher concentration of predators. I also shifted the concentration of predators toward the center of the interval, so that $u^0(x) = 0.3 \exp(-50(x-0.2)^2)$ and $v^0(x) = \exp(-50(x-0.5)^2)$ (See Figure 2). Other parameter values are a = 0.7, b = 0.3, and D = 0.5.



Figure 3: Numerical approximations of the modified Lotka-Volterra model with a smaller population of prey and a larger population of predators. Figure 3a shows the distribution of the population of the prey species at various time steps, and Figure 3b shows the distribution of population of the predator species at various time steps.

The results shown here are consistent with varying values of a, b, and D. The presence of a dispersal term in the model has a stabilizing effect, and this result has been proven in several variations of the Lotka-Volterra equations (including [2, 5, 10]). Increasing D causes the populations to achieve a uniform distribution more quickly. After the populations are (nearly) uniform, the two populations will then begin to converge to the stable solution (u, v) = (b, 1-b).

In summary, the Lotka-Volterra equations have historically played an important role in modeling predator-prey dynamics. Though the non-trivial solution to the system is potentially unrealistic, it can be easily modified to more closely mimic what happens in nature. Specifically, one such modification is the addition of a diffusion term which causes the solution to be stable. The model itself can be numerically solved using finite difference methods. The Matlab code provided in the appendix can be easily modified to reflect other changes in the model as it suits the user.

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Appendix

Matlab code used to numerically solve the Lotka-Volterra model with diffusion by using a forward finite difference scheme

```
% Forward Method
clear;
L = 1; %total length of spatial interval
T = 1; %total length of time interval
% Parameters needed to solve the equation within the explicit method
maxk = 10000; % Number of time steps
dt = T/maxk;
nx = 50; % Number of space steps
dx = L/nx;
a = .7;
b = .3;
nu=dt/(dx*dx);
k=50; %parameter in population normal distributions
D =.5; % diffusion constant
% Initial distributions
for j = 1:nx+1
                 x(j) = (j-1) * dx;
                 %u(j,1) =.8; %sin(pi*x(j)); %other initial distributions
                u(j, 1) = 0.8 \exp(-k.*((x(j)-0.2)).^2);
                 %v(j,1) =.4;
                 %v(j,1)=.25+.5.*sin(pi*x(j));
                 v(j, 1) = 0.3 + exp(-k.*((x(j)-0.7)).^2);
end
% Implementation of the forward method
for n=1:maxk % Time Loop
                 j = 1; %left-hand boundary
                                 u(j,n+1) = D*nu.*(u(j,n)+u(j+1,n)-2.*u(j,n))+ dt*(u(j,n)).*(1-u(j,n)-2.*u(j,n))+ dt*(u(j,n)).*(1-u(j,n)-2.*u(j,n))+ dt*(u(j,n)).*(1-u(j,n)-2.*u(j,n))+ dt*(u(j,n))+ dt*(u(j,n)))+ dt*(u(j,n))+ dt*(u(j
v(j,n))+u(j,n);
                                 v(j,n+1) = nu.*(v(j,n)+v(j+1,n)-2.*v(j,n)) + dt*a.*(v(j,n)).*(u(j,n)-
b) + v(j, n);
                 for j=2:nx; % Space Loop
                                 u(j,n+1) = D*nu.*(u(j-1,n)+u(j+1,n)-2.*u(j,n))+ dt*(u(j,n)).*(1-
u(j,n) - v(j,n) + u(j,n);
                                 v(j,n+1) = nu.*(v(j-1,n)+v(j+1,n)-2.*v(j,n)) +
dt*a.*(v(j,n)).*(u(j,n)-b) + v(j,n);
                 end
                 %right-hand boundary
                 j = nx+1;
                                 u(j,n+1) = D*nu.*(u(j-1,n)+u(j,n)-2.*u(j,n))+ dt*(u(j,n)).*(1-u(j,n)-2.*u(j,n))+ dt*(u(j,n)).*(1-u(j,n)-2.*u(j,n))+ dt*(u(j,n)).*(1-u(j,n)-2.*u(j,n))+ dt*(u(j,n)))+ dt*(u(j,n))+ dt*(u(j
v(j,n) + u(j,n);
                                  v(j,n+1) = nu.*(v(j-1,n)+v(j,n)-2.*v(j,n)) + dt*a.*(v(j,n)).*(u(j,n)-1)
b) + v(j,n);
end
```

```
% Graphical representation of the temperature at different selected times
figure(1)
plot(x,u(:,1),'-',x,u(:,300),'--',x,u(:,1000),':',x,u(:,10000),'-.')
axis([0 1 0 1]) %specifies limits of axes (0, 1) x (0, 1)
title('Prey distributions at various time steps')
xlabel('X')
ylabel('X')
ylabel('u')
figure(2)
plot(x,v(:,1),'-',x,v(:,300),'--',x,v(:,1000),':',x,v(:,10000),'-.')
axis([0 1 0 1])
title('Predator distribution at various time steps')
xlabel('X')
ylabel('X')
```